ALGUNES EINES MATEMÀTIQUES DE LA GEOMÀTICA

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15a Trobada Matemàtica
Societat Catalana de Matemàtiques
Barcelona, 2012-05-25

Die Geodäsie ist im allgemeinen ein glänzendes Beispiel dafür,
was man mit der Mathematik in den Anwendungen
machen kann und wie man es machen soll.

- Felix Klein (1902, 1849-1925)

Borre,K. (ed.), 2006. The mathematical foundations of geodesy: selected papers of Torben Krarup. Springer.

UNA RELACIÓ “win-win”

- durant un quan temps, la geodèsia va ésser un dels principals estimuladors de la matemàtica

- la geodèsia ha estat un beneficiari primerenc de la matemàtica

present

- la geomàtica s'enfronta a reptes científics enormes que

  no tindran solució sense abstraccions matemàtiques potents i simples

Einfachheit ist das Resultat der Reife.

- Friedrich Schiller (1759-1805)

WHAT IS GEOMATICS?

the art, science and technology of dealing with georeferenced (spatially tagged) information: acquisition, orientation, interpretation, storage and dissemination.

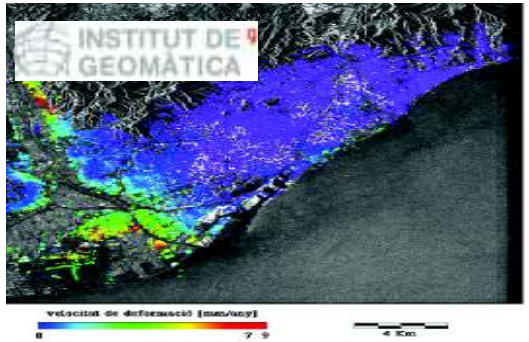
the disciplines of geomatics

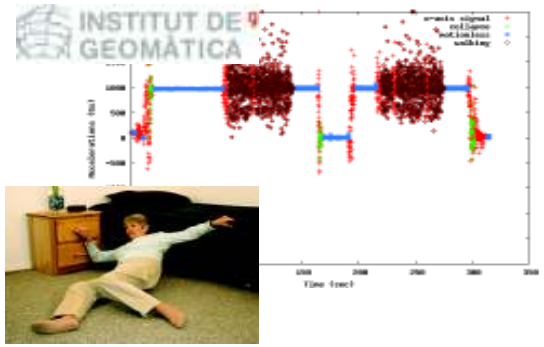
- geodesy (and navigation)
- remote sensing (including photogrammetry)
- cartography (geo-modeling and geo-visualization)

HAVE YOU EVER MET A “GEOMATICIAN”?

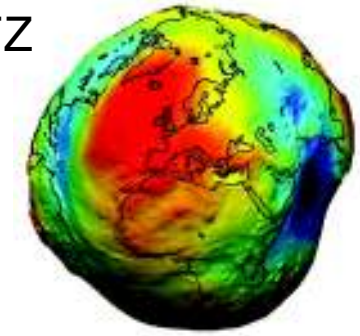
Google

maps





GFZ



GEODESY

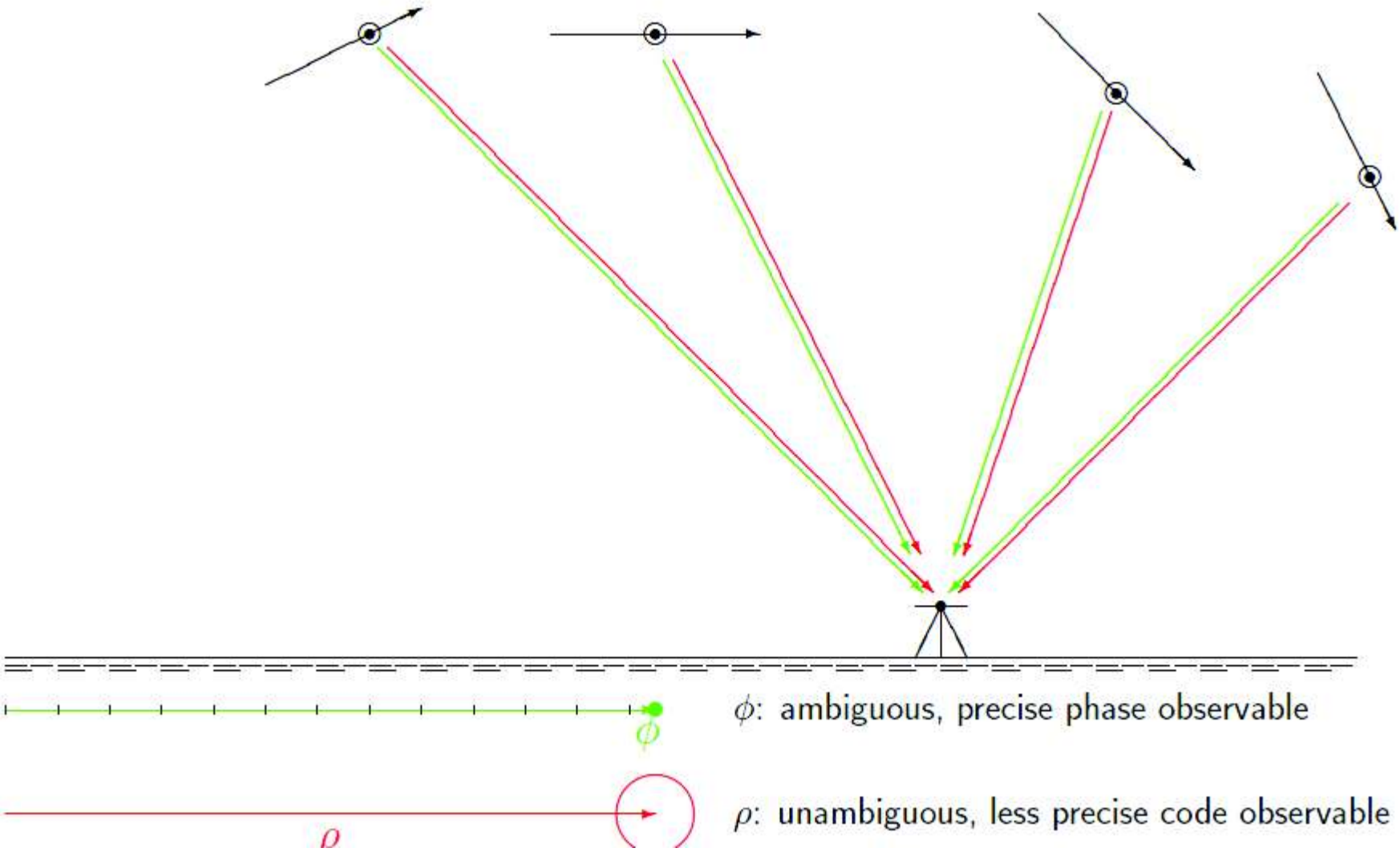
is the scientific discipline that deals with the measurement and representation of the Earth, including its gravitational field, in a three-dimensional time-varying space.

the sub-disciplines of geodesy

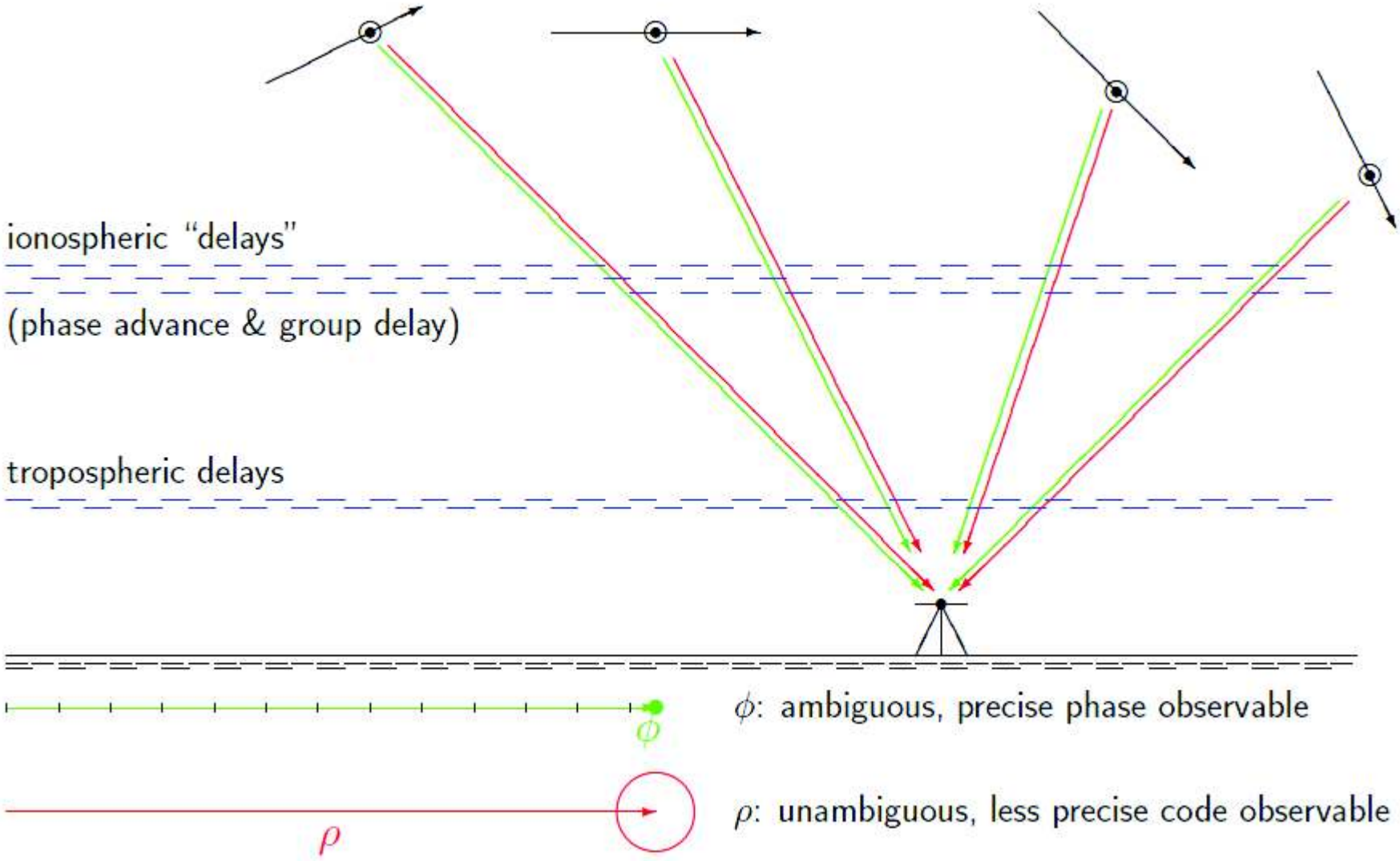
- positioning (broad sense) [real-time positioning is navigation]
- gravity field determination

Wikipedia, accessed 2012-05-24.

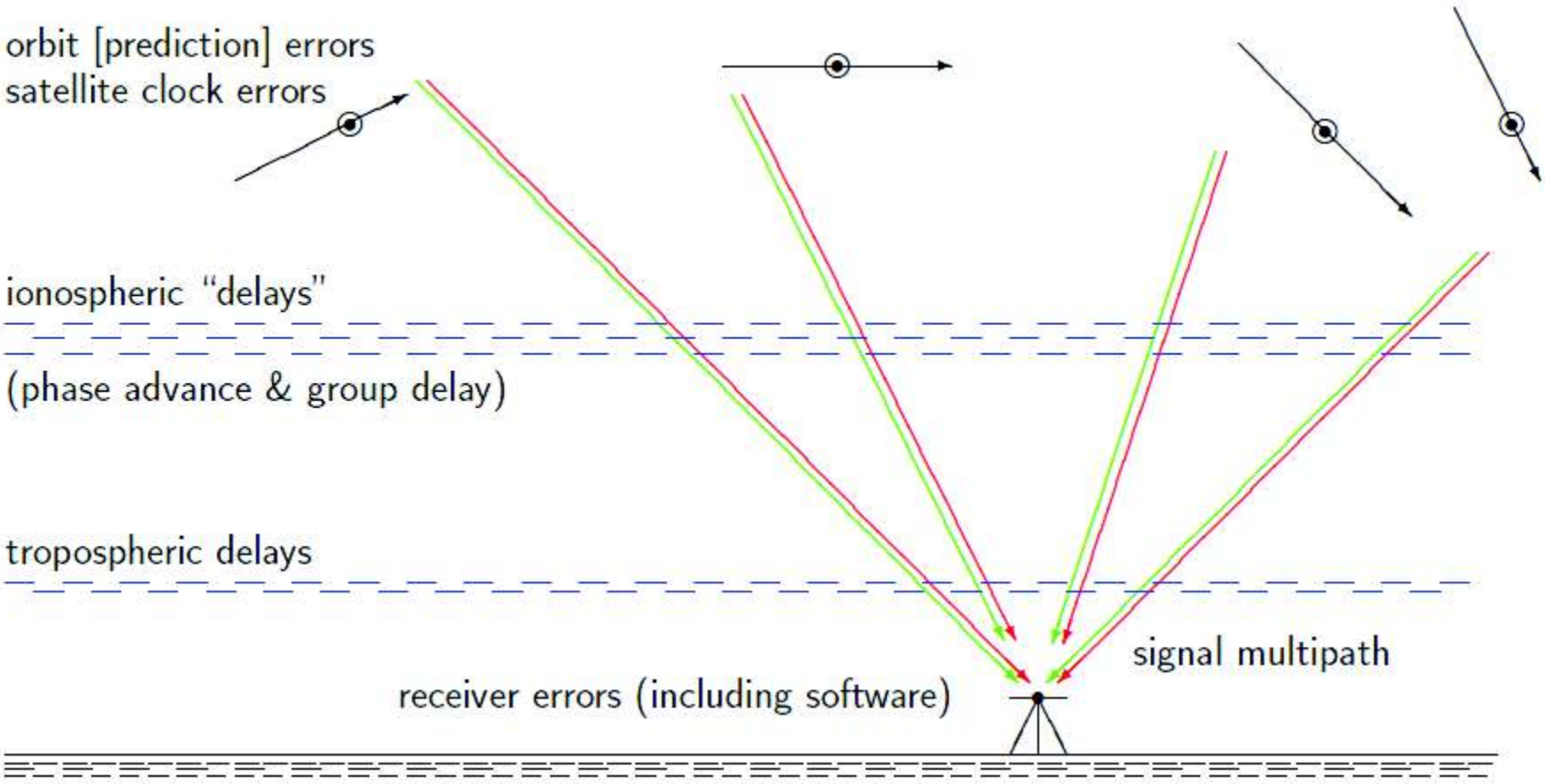
Problem: POSITIONING WITH GPS/GNSS



POSITIONING WITH GPS/GNSS



POSITIONING WITH GPS/GNSS



FROM

phase observation equations

$$\begin{aligned}\lambda_1 \cdot \Phi_{1r}^s &= \rho_r^s + \tau_r^s - I_r^s f_1^{-2} \quad + \lambda_1 N_{1r}^s \ , \\ \lambda_2 \cdot \Phi_{2r}^s &= \rho_r^s + \tau_r^s - I_r^s f_2^{-2} \quad + \lambda_2 N_{2r}^s \ ,\end{aligned}$$

code observation equations

$$\begin{aligned}C_{1r}^s &= \rho_r^s + \tau_r^s + I_r^s f_1^{-2} \ , \\ P_{2r}^s &= \rho_r^s + \tau_r^s + I_r^s f_2^{-2} \ .\end{aligned}$$

with

$$\rho_r^s = \sqrt{(X_r - X^s)^2 + (Y_r - Y^s)^2 + (Z_r - Z^s)^2}$$

TO

phase observation equations

$$\begin{aligned}
\lambda_1 \cdot \Phi_{1r}^s &= \rho_r^s + c\,(\delta t^s - \delta T_r) + \tau_r^s - I_r^s f_1^{-2} \;+\lambda_1\,(N_{1r}^s + \varphi_r(t_0) - \varphi^s(t_0)) \\
\lambda_2 \cdot \Phi_{2r}^s &= \rho_r^s + c\,(\delta t^s - \delta T_r) + \tau_r^s - I_r^s f_2^{-2} \;+\lambda_2\,(N_{2r}^s + \varphi_r(t_0) - \varphi^s(t_0)) \;-d\rho_r^s \;+b_1,
\end{aligned}$$

code observation equations

$$\begin{aligned}
C_{1r}^s &= \rho_r^s + c\,(\delta t^s - \delta T_r) + \tau_r^s + I_r^s f_1^{-2} \qquad\qquad\qquad\qquad\qquad\qquad +b_2, \\
P_{2r}^s &= \rho_r^s + c\,(\delta t^s - \delta T_r) + \tau_r^s + I_r^s f_2^{-2} \qquad\qquad\qquad\qquad\qquad -d\rho_r^s \;+b_3.
\end{aligned}$$

with

$$\rho_r^s = \sqrt{(X_r - X^s - \delta X^s)^2 + (Y_r - Y^s - \delta Y^s)^2 + (Z_r - Z^s - \delta Z^s)^2}$$

CHALLENGES AND TECHNIQUES

solution of a

- mixed real-integer least-squares optimization problem

  LAMBDA method

- stochastic equation

  unknowns: random variables & stochastic processes

essentially unsolved problem of modeling...

$$\delta\dot{T}_r = w \qquad w \sim N(0, \sigma_r)$$

$$\delta\dot{T}_r = -\frac{1}{\tau}\delta T_r + w \qquad w \sim N(0, \sigma_r)$$

Problem 2: POSITIONING WITH INS

inertial measurements

- 3 linear accelerations
- 3 angular rates

+

gravity field model

→ **tPVA solution**

POSITIONING WITH INS

$$
\begin{aligned}
\dot{x}^e &= v^e \\
\dot{v}^e &= R_b^e f^b - 2\Omega_{ie}^e v^e + g^e(x^e) \\
\dot{R}_b^e &= R_b^e \left(\Omega_{ei}^b + \Omega_{ib}^b\right)
\end{aligned}
$$

POSITIONING WITH INS

$$
\begin{aligned}
\dot{x}^e &= v^e \\
\dot{v}^e &= R_b^e \left(R_{wa}^b (1 + S_a)(f^b + w_f^b - b_a) \right) - 2\Omega_{ie}^e v^e + g^e(x^e) \\
\dot{R}_b^e &= R_b^e \left(\Omega_{ei}^b + \left[(R_{wo}^b (1 + S_o)(\omega_{ib}^b + w_{\omega}{}_{ib}^b - b_o)) \times \right] \right) \\
\dot{b}_o &= -\beta_b \, b_o + w_{b_o} \\
\dot{b}_a &= -\alpha_b \, b_a + w_{b_a} \\
\dot{s}_o &= -\beta_s \, s_o + w_{s_o} \\
\dot{s}_a &= -\alpha_s \, s_a + w_{s_a} \\
\dot{m}_o &= -\beta_m \, m_o + w_{m_o} \\
\dot{m}_a &= -\alpha_m \, m_a + w_{m_a}
\end{aligned}
$$

calibration unknowns

+

noise

Stochastic Differential Equation $\longrightarrow$ **DRIFTS!**

POSITIONING WITH INS/GNSS

$$\begin{cases} \mathbf{x}(t) \sim \mathrm{SDS}(\hat{\mathbf{x}}^-(t), \mathbf{C}_{\mathbf{x}}^-(t)) \\ \ell(t) = \mathbf{H}(t)\mathbf{x}(t) - \mathbf{v}(t), \qquad \mathbf{v}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\mathbf{v}}) \text{ (measurement)} \end{cases}$$

$$\begin{cases} \mathbf{x}_k = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1}^-, \qquad \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k) \text{ (system noise)} \\ \ell_k = \mathbf{H}_k\mathbf{x}_k - \mathbf{v}_k, \qquad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) \text{ (measurement noise)} \end{cases}$$

POSITIONING WITH INS/GNSS

prediction step (many)

$$\begin{cases} \hat{\mathbf{x}}_k^- &= \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1} \\ P_k^- &= \Phi(t_k, t_{k-1})P_{k-1}\Phi^T(t_k, t_{k-1}) + (t_k - t_{k-1})G_k Q_k G_k^T \end{cases}$$

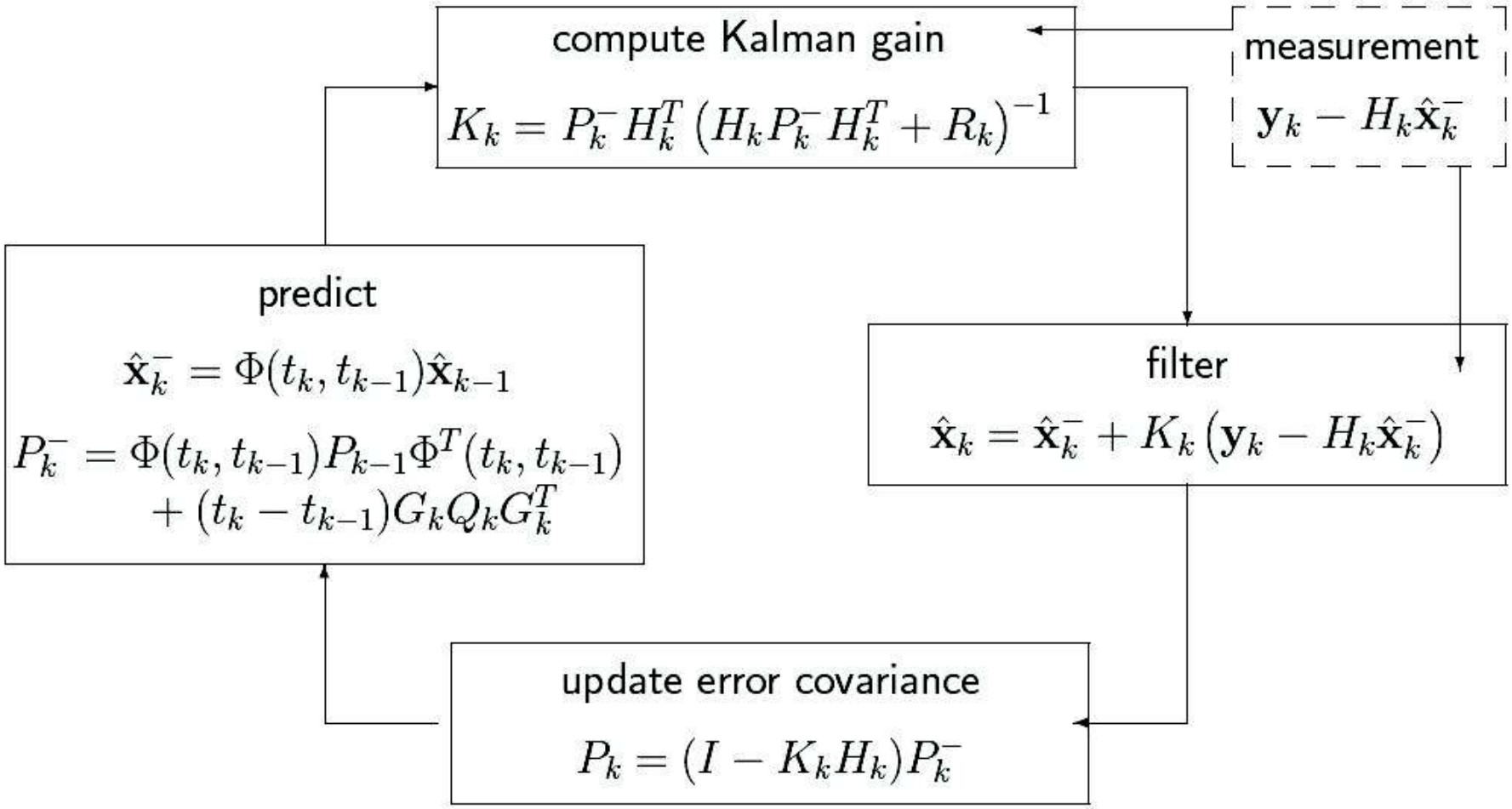
update step (less)

$$\begin{cases} \hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_k^- + K_k\left(\mathbf{y}_k - H_k\hat{\mathbf{x}}_k^-\right) \\ P_k &= (I - K_k H_k)P_k^-(I - K_k H_k)^T + K_k R_k K_k^T \end{cases}$$

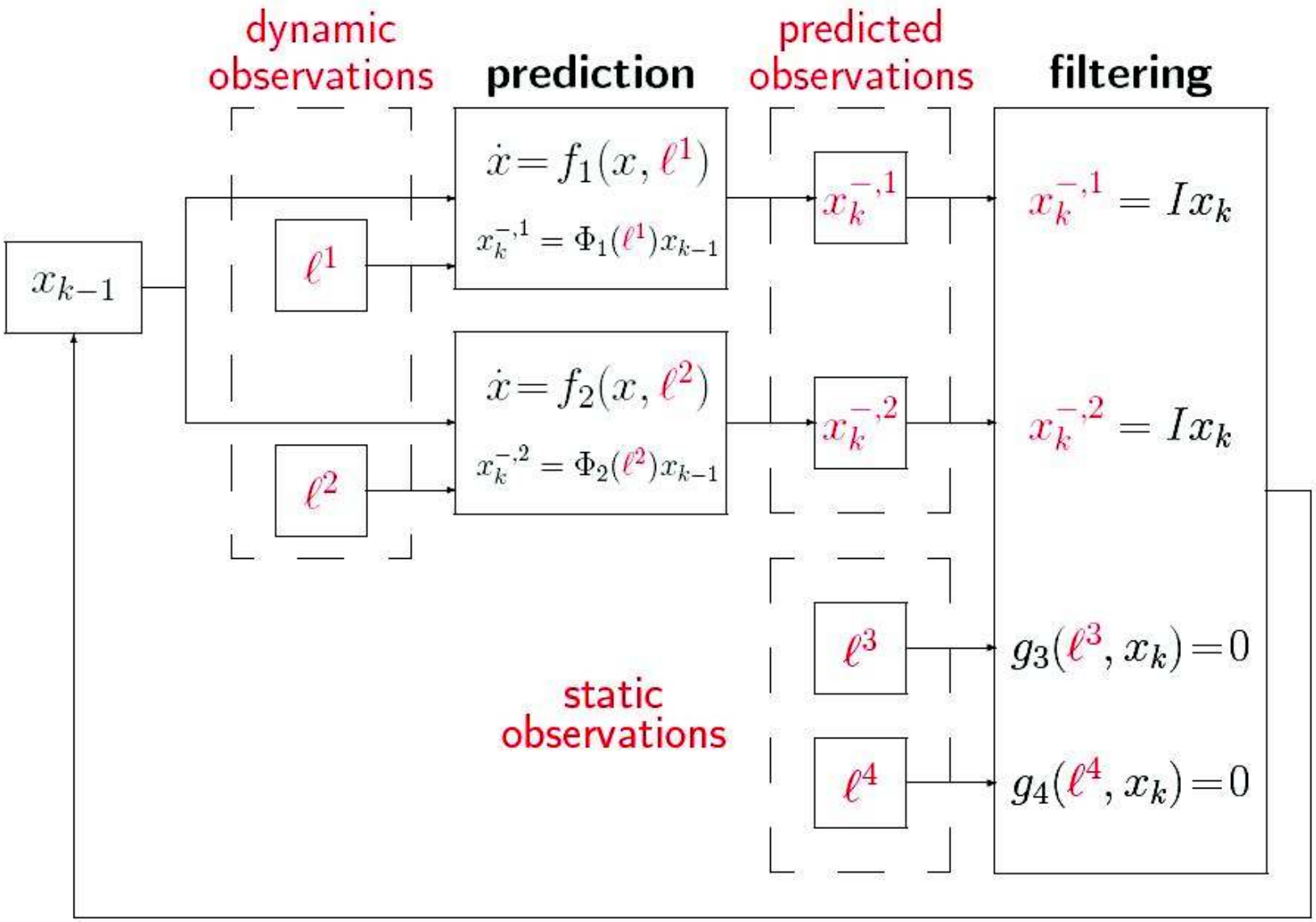
alternatives:

- òbvia! solució numèrica per a E(x)
- equació de Ricatti per a C(x)

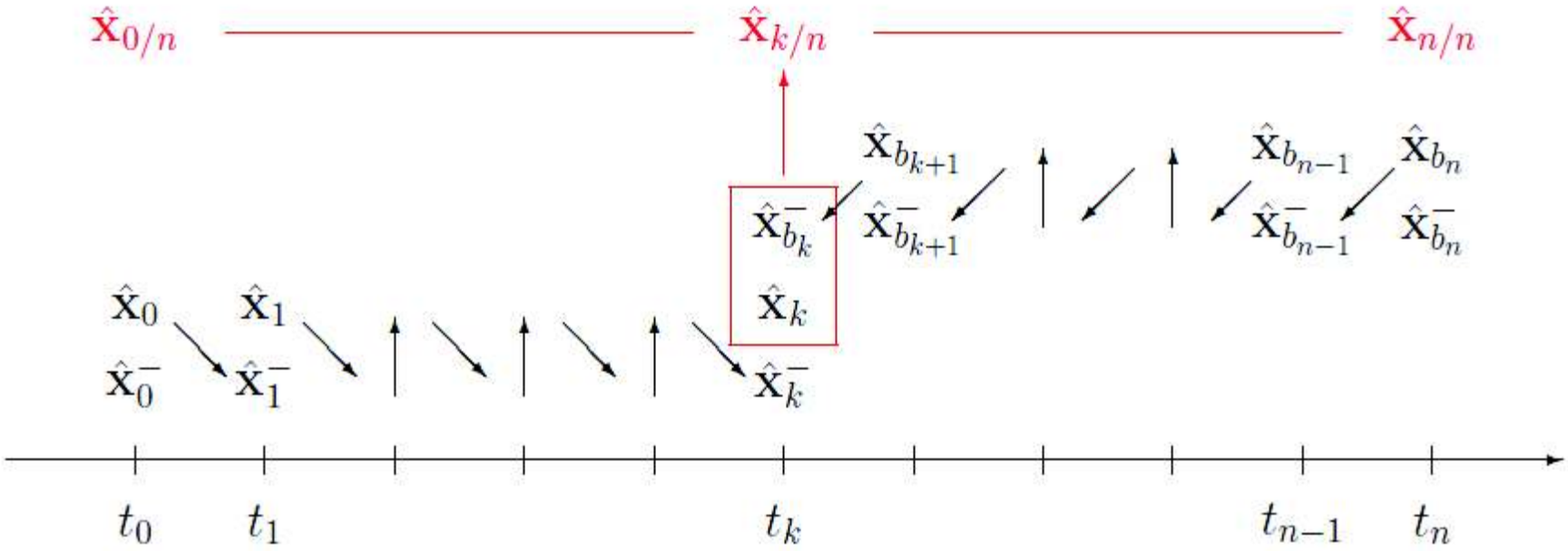
CLASSICAL “KALMAN FILTER”



EQUIVALENT LEAST-SQUARES FORMULATION

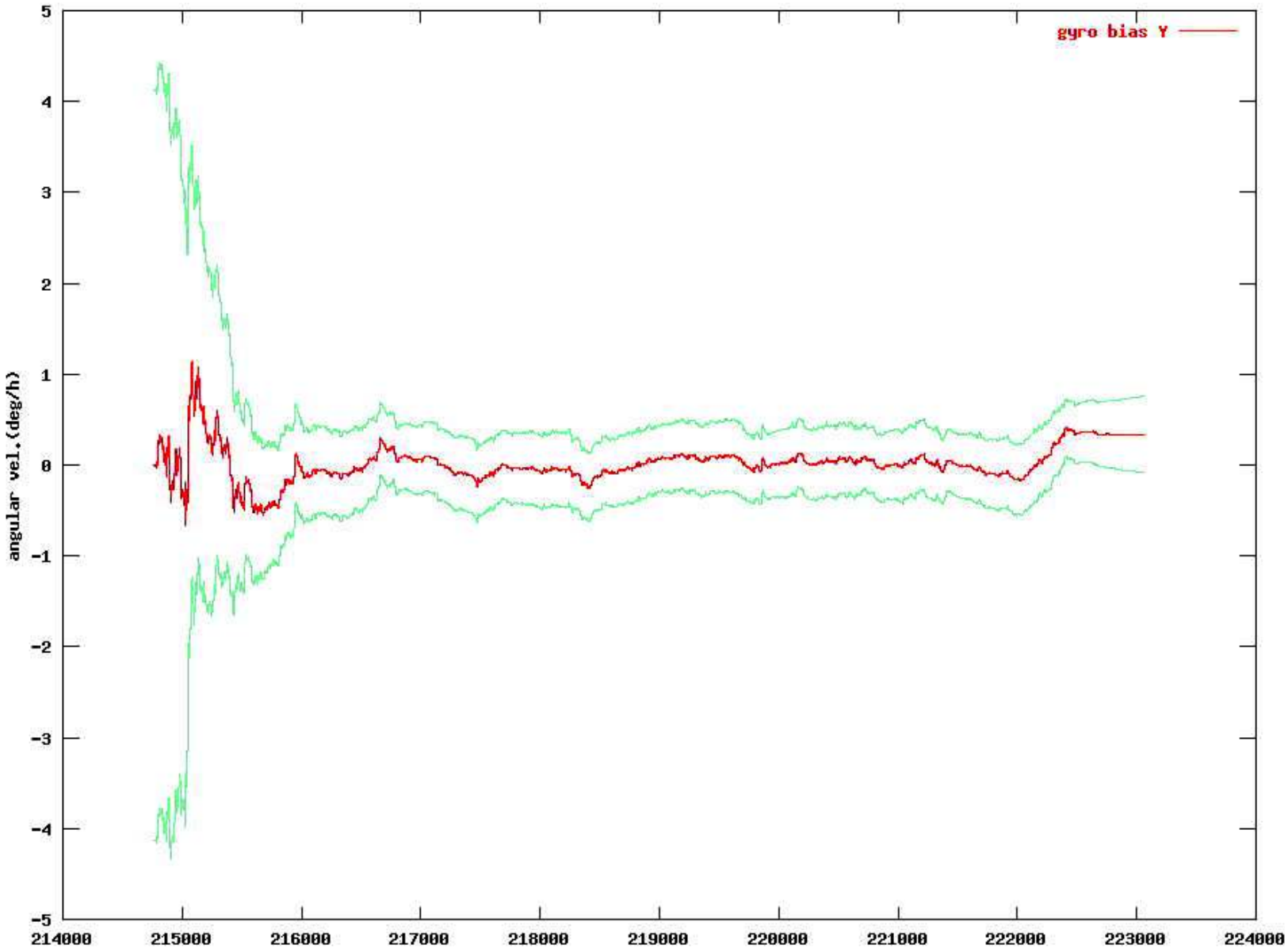


KALMAN FILTERING & SMOOTHING

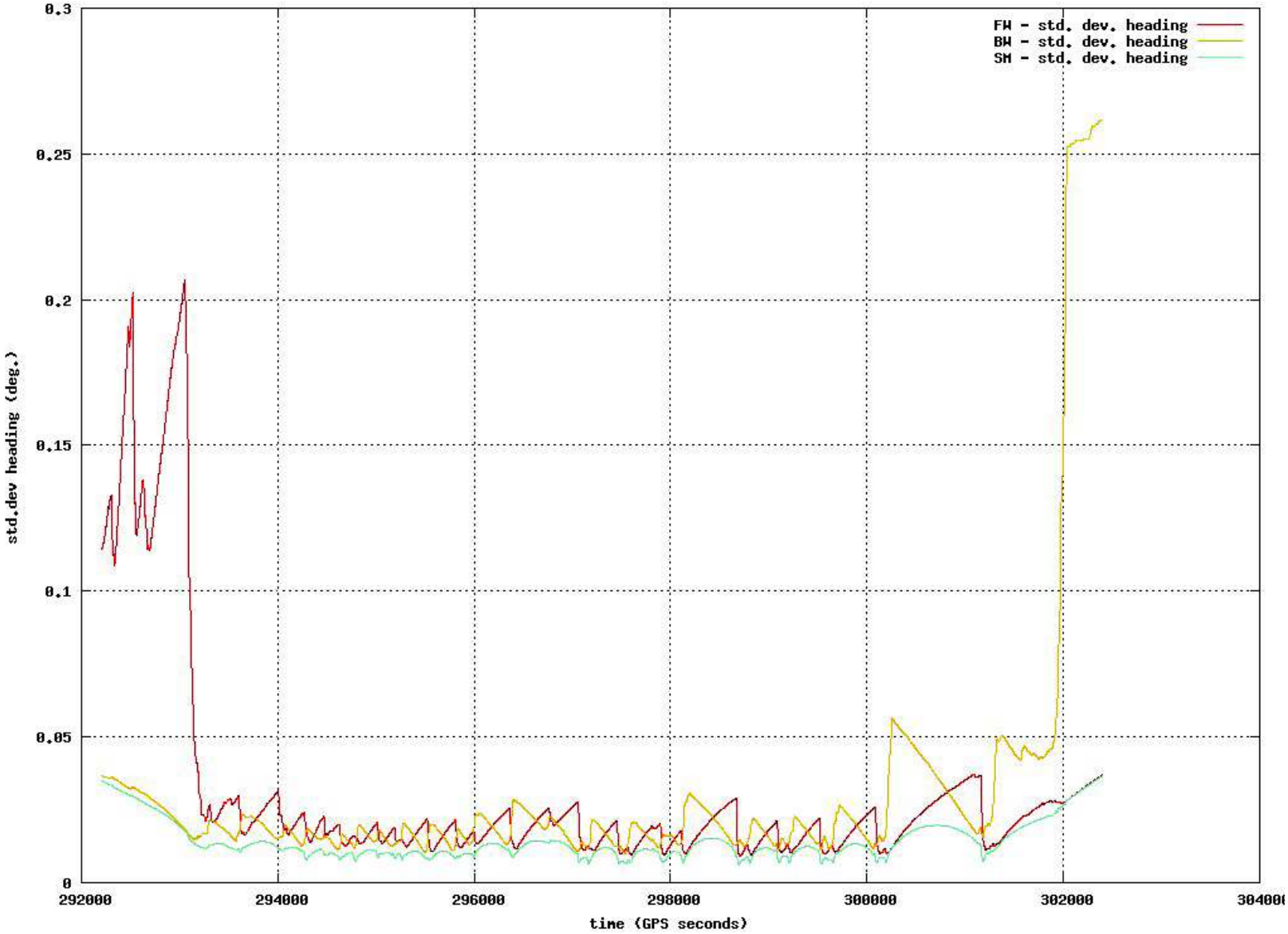


$\hat{\mathbf{x}}_k^-$: predicted [forward] at t_k	$\hat{\mathbf{x}}_k$: filtered [forward] at t_k
$\hat{\mathbf{x}}_{b_k}^-$: predicted [backward] at t_k	$\hat{\mathbf{x}}_{b_k}$: filtered [backward] at t_k
$\hat{\mathbf{x}}_{k/n}$: smoothed at t_k	$\hat{\mathbf{x}}_{k/n} = \hat{\mathbf{x}}_n$

w_y BIAS CALIBRATION PATH



COVARIANCE MATRIX EVOLUTION



STOCHASTIC PROCESS ERROR MODELING... largely unsolved

Alan Variance (of a path)

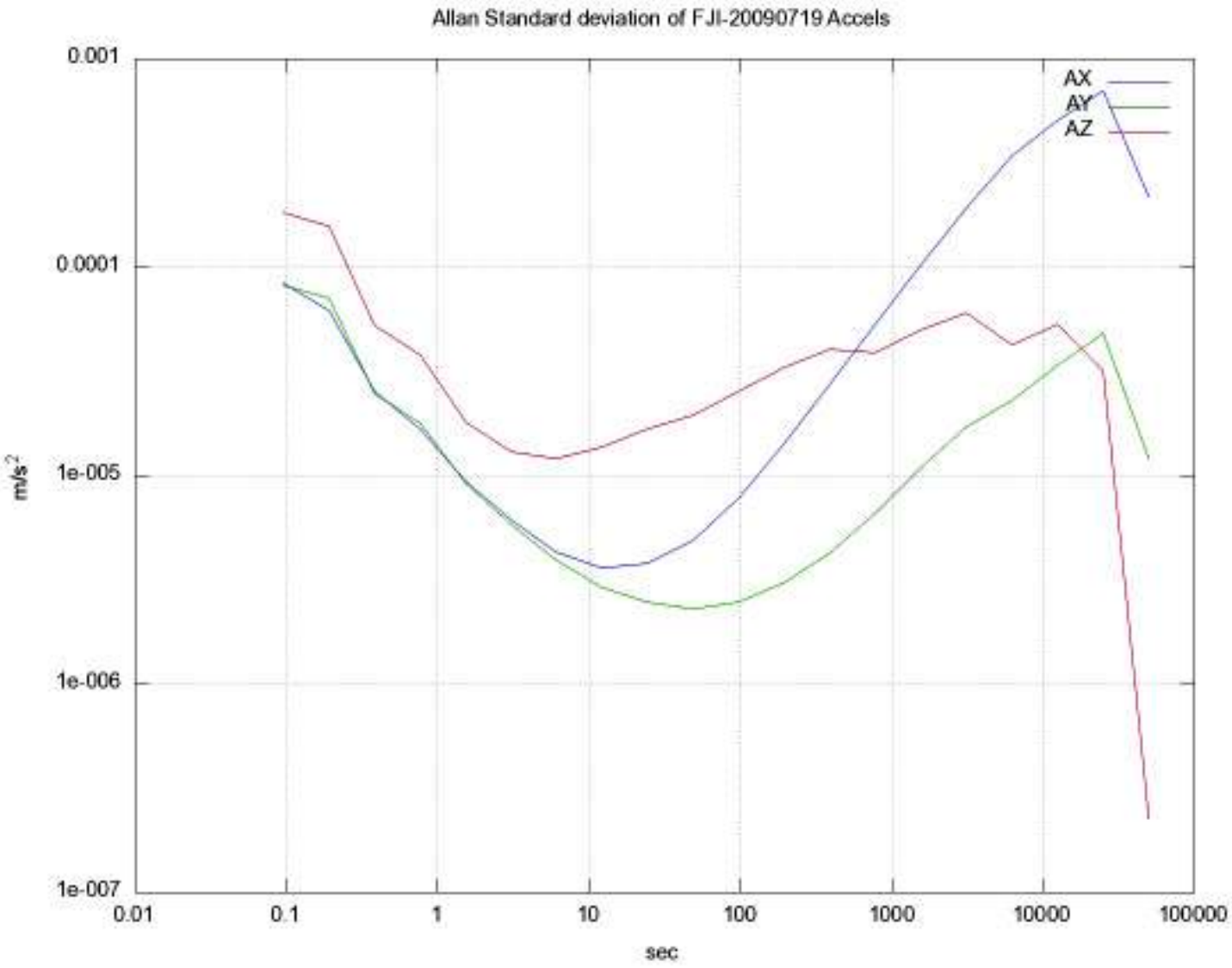
$$\sigma_\Omega^2(\tau) = \frac{1}{2}\left\langle \left[\bar{\Omega}_{k+1}(\tau) - \bar{\Omega}_k(\tau)\right]^2 \right\rangle$$

$$\bar{\Omega}_k(\tau) = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} \Omega(t)dt$$

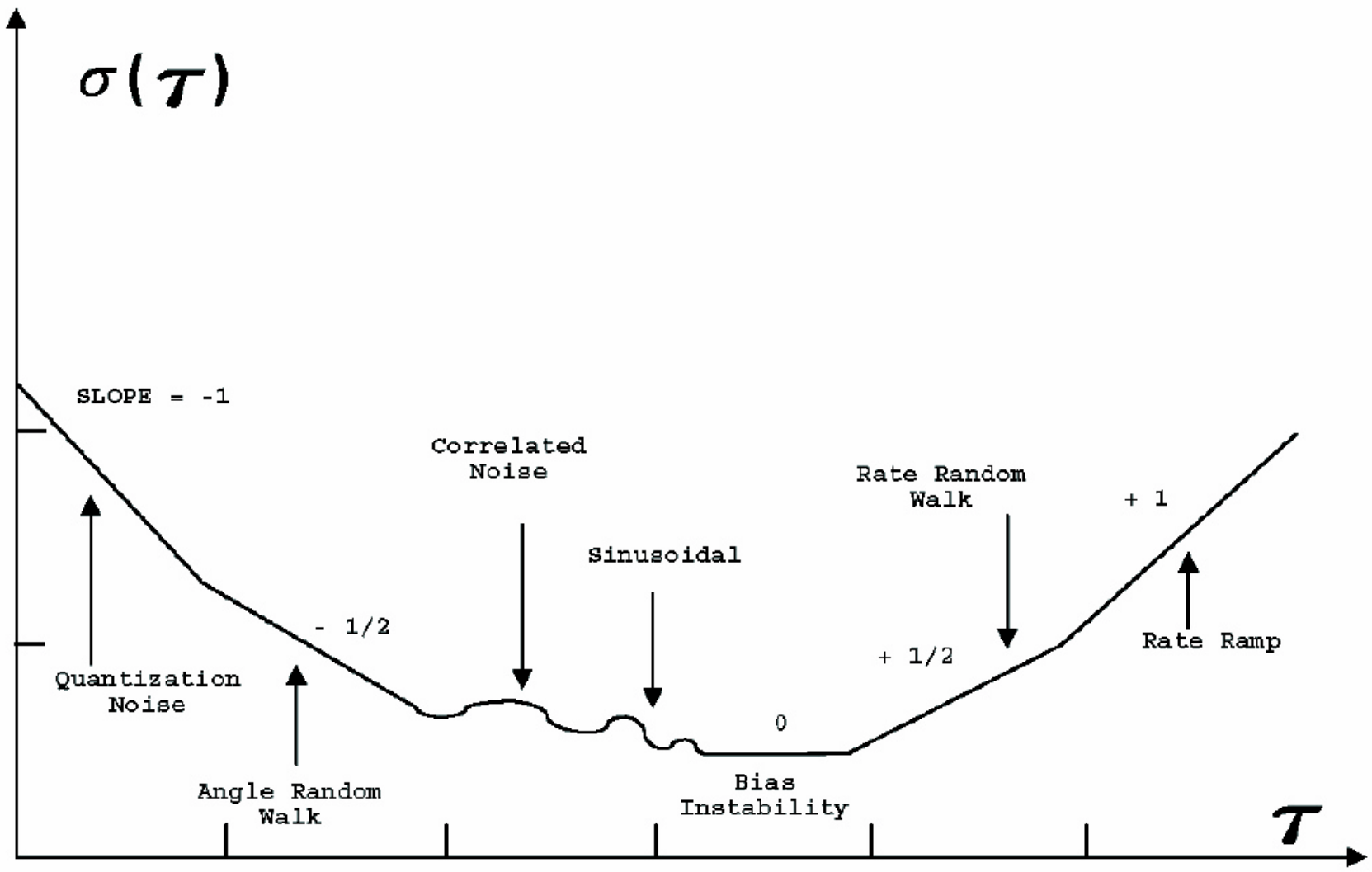
results in

$$\sigma_\Omega^2(\tau) = \frac{1}{2(N-2n)} \sum_{k=1}^{N-2n} \left[\bar{\Omega}_{k+1}(\tau) - \bar{\Omega}_k(\tau)\right]^2$$

ALAN STANDARD DEVIATION OF AN ACCELEROMETER



ALAN STANDARD DEVIATION



APLICACIÓ: PROJECTE CLOSE-SEARCH / FP7 i GSA

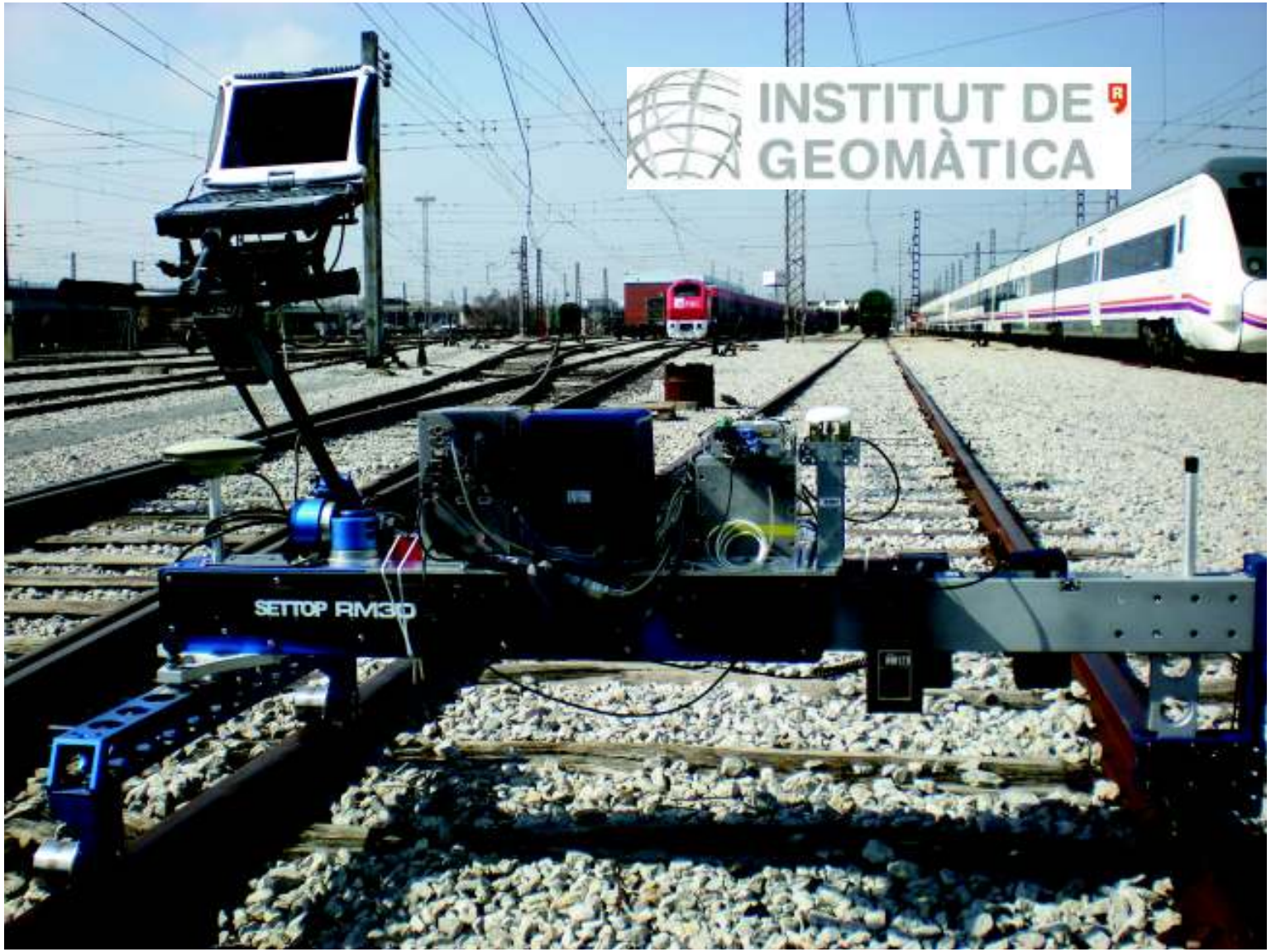
Projecte CLOSE-SEARCH

posicionament
navegació
fotogrametria
teledetecció
geoinformació

HW & SW

DGPC
Bombers CAT

PROJECTE DECIVEL / PROGRAMA PROVA'T (GenCat)



PROJECTE SARVANT

SARVANT: miniradar, InSAR, bandes X i P	**- OrbiSat**
- VANT	**- Aeroálcool**
- guiatge i control	**- AGX**
- orientació precisa i navegació redundant	**- OrbiSat / IG**

pes total: 140 kg

càrrega: 35 kg

ori&nav: 8 kg

màx t- vol: 20 h

velocitat: 200 km/h

SAR banda X
SAR banda P
òptic 2012

SAFE NAVIGATION

regression diagnostics and robust estimators

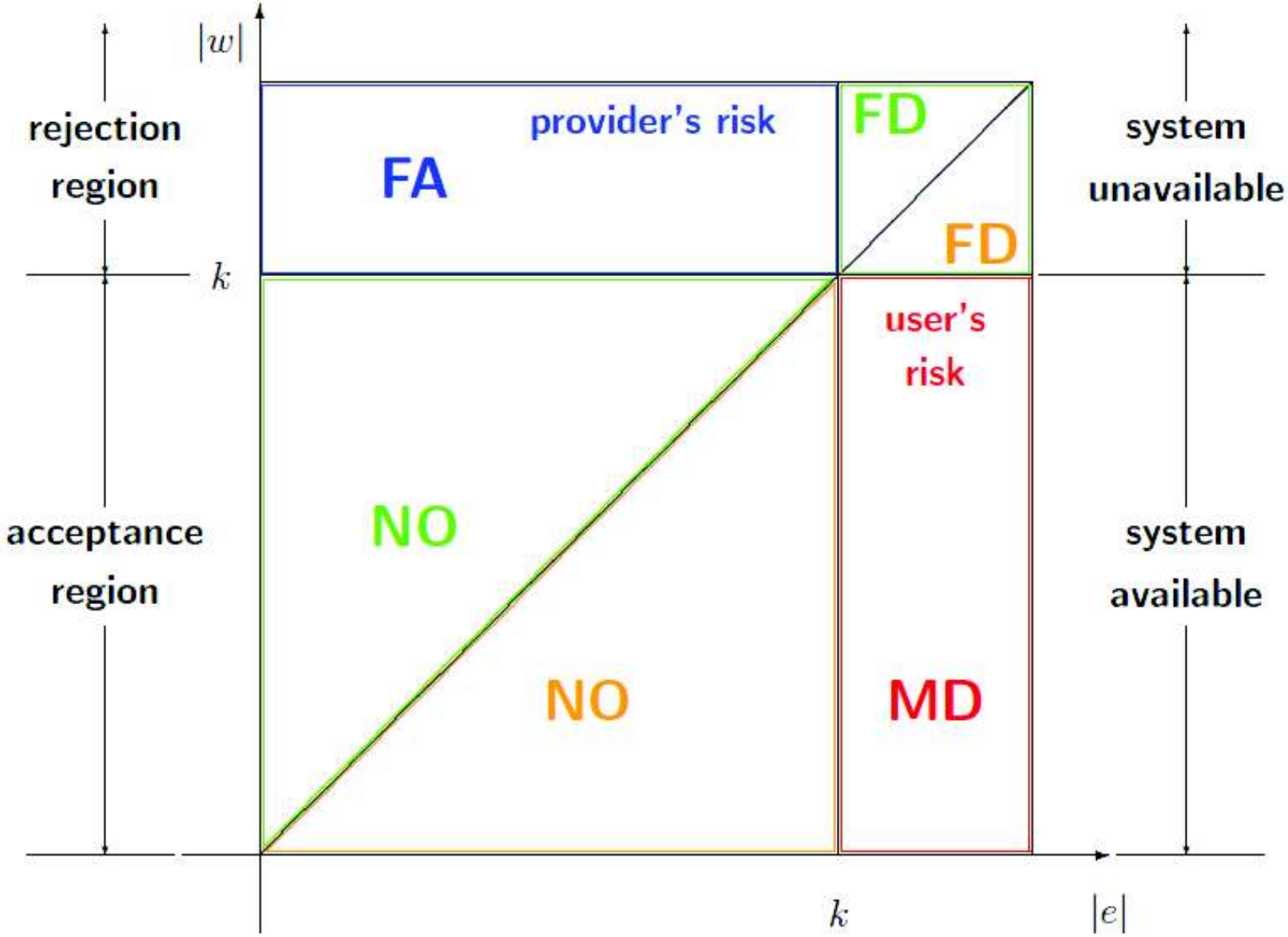
It ain't what you don't know that gets you into trouble.

It's what you know for sure that just ain't so.

- Mark Twain (1835-1910)

(Riverboat pilot on the Mississippi River.)

SAFE NAVIGATION: STANFORD PLOT



SAFE NAVIGATION

FT: the tree of all possible non-nominal error sources of the navigation sensors of a navigation system.

The leaves of FT constitute the set FE of failure events, $\mathcal{F}$.

$$\mathcal{F} = \{e_1, \ldots, e_n\} \qquad \mathcal{P} = \{p_1, \ldots, p_n\}$$

Probability of the system performing under nominal conditions:

$$\Pi_{i=1}^{n}(1 - p_i), \quad p_i = p(e_i)$$

type II error probability

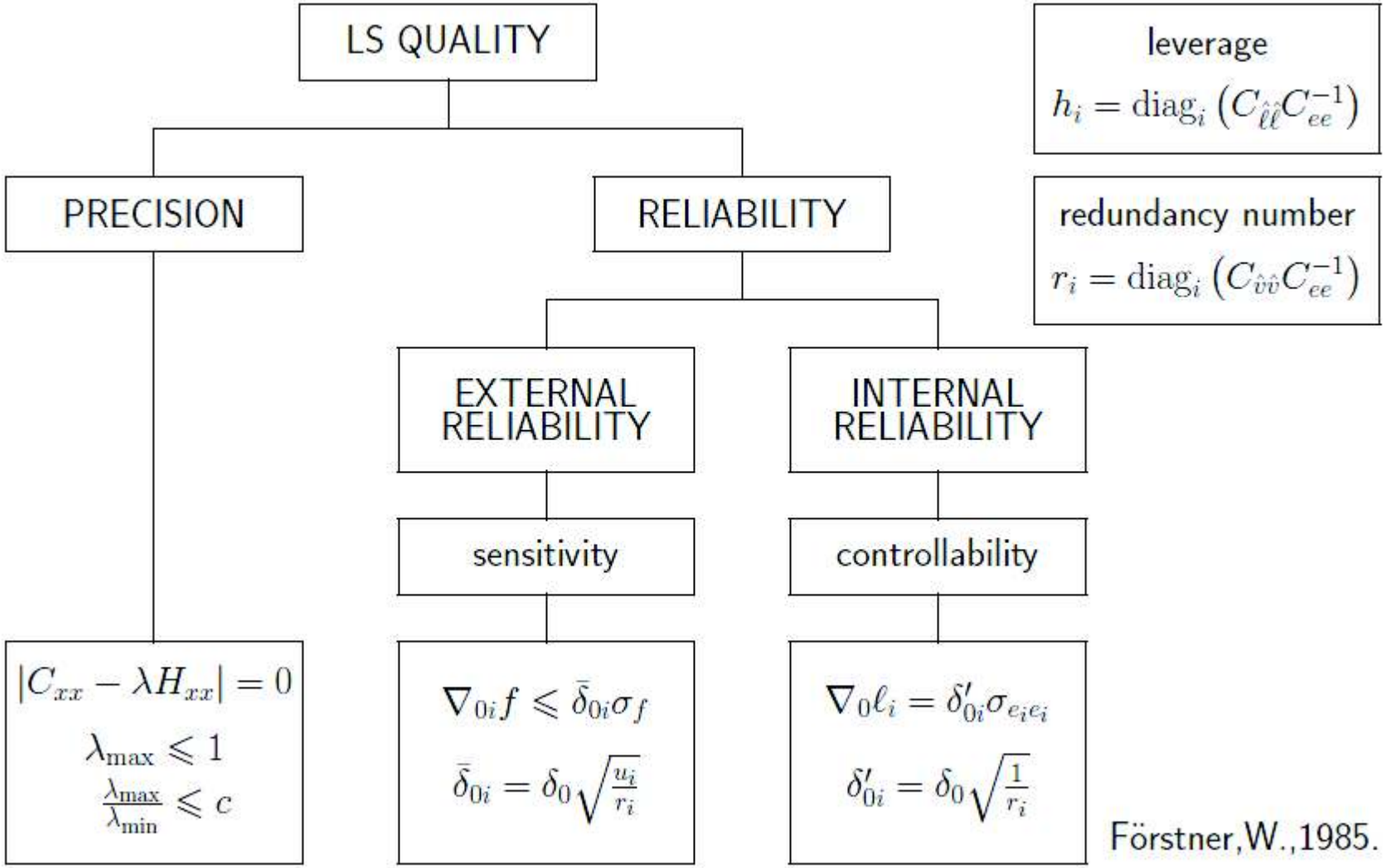
consumer's risk

The integrity risk (IR) can now be computed as:

$$IR = a_0 + \sum_{i=1}^{n} p_i \cdot (1 - \beta_{0i})$$

risk in the absence of outliers

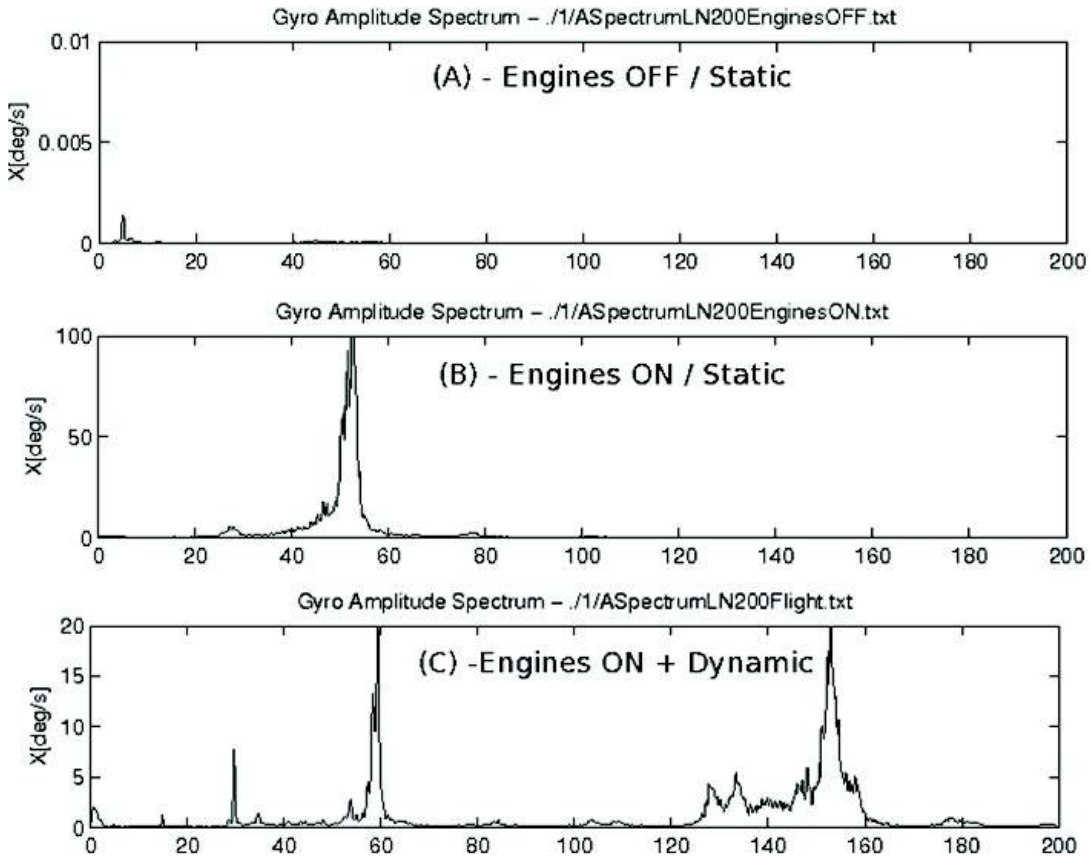
GEODETIC APPROACH TO SAFE NAVIGATION



Förstner,W.,1985.

OTHER CHALLENGES OF CLOSE-SEARCH

- vibrations



- automatic image interpretation
- use of low-cost sensors: calibration models

  0.01 deg/h – degs/s

SIMULTANEOUS LOCALIZATION AND MAPPING (SLAM)

Stanley: University of Stanford

DARPA: Grand Challenge

TERRESTRIAL NAVIGATION

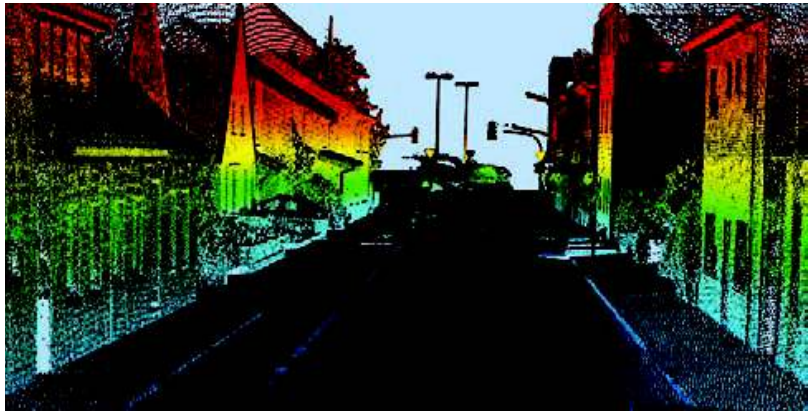
FROM MASS-MARKET

Google Earth

500 million users

maps of the present

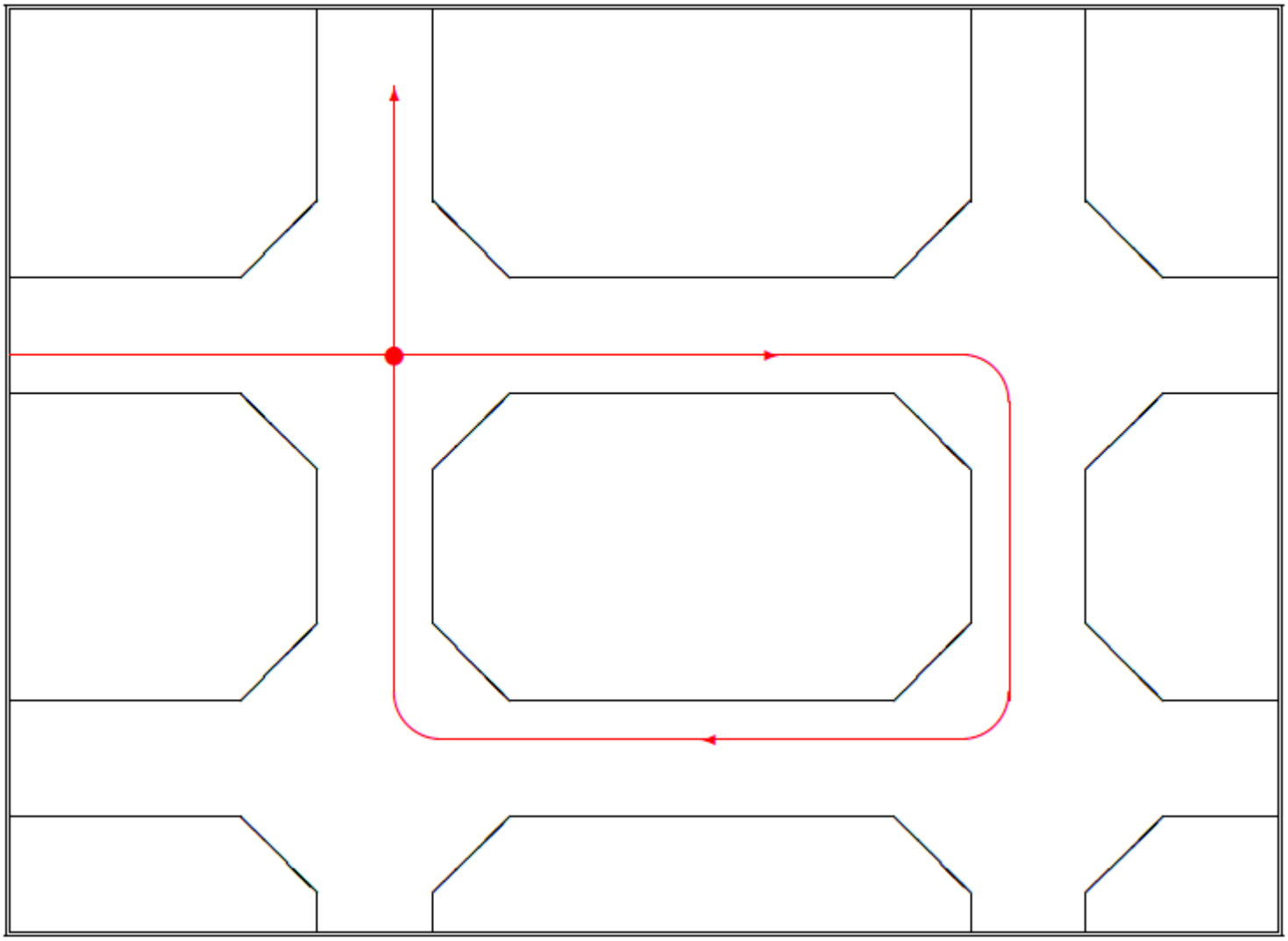
TO PROFESSIONAL



maps of the future

DYNAMIC NETWORKS

GÖTTERDÄMMERUNG OVER KALMAN-FILTER FOR MAPPING

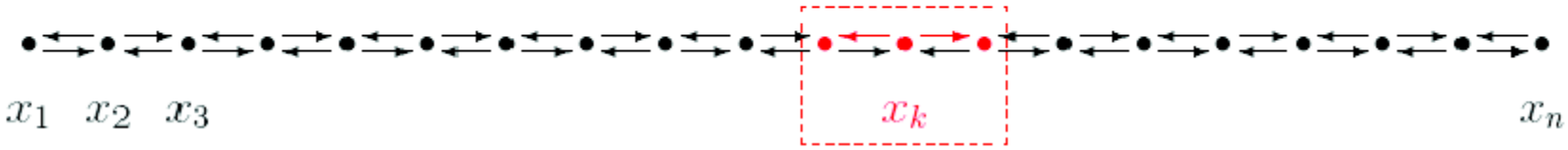


DYNAMIC NETWORKS

- solution of a stochastic differential equation (SDE) by diff. methods
- not an initial value problem, not a boundary value problem

• INS mechanization equations - Difference Equation model

$$
\begin{aligned}
x^e_{k+1} - x^e_{k-1} &= \delta t \cdot v^e_k \\
v^e_{k+1} - v^e_{k-1} &= \delta t \cdot \left([R^e_b]_k f^b{}_k - 2\Omega^e_{ie} v^e_k + g^e(x^e_k) \right) \\
[R^e_b]_{k+1} - [R^e_b]_{k-1} &= \delta t \cdot [R^e_b]_k \left([\Omega^b_{ei}]_k + [\Omega^b_{ib}]_k \right)
\end{aligned}
$$



unknowns: 4 h x 3600 s/h x 500 Hz x (3 + 3 + 4 + 6 + 6 + (≈) 20)

≈ 302.400.000

REMOTE SENSING & PHOTOGRAMMETRY

is the art, science and technology of obtaining reliable information about physical objects and the environment through the process of recording, measuring, and interpreting photographic images and patterns of electromagnetic radiant energy and other phenomena.

the sub-disciplines of remote sensing & photogrammetry

- photogrammetry (dominated by a geometry oriented approach)
- remote sensing (dominated by a physical oriented approach)

(not valid for long)

McGlone,J.C. (ed.),2004. Manual of Photogrammetry (5th edition). American Society for Photogrammetry and Remote Sensing, Bethesda, MA, US.

THE FOUR PARADIGMS OF REMOTE SENSING: 1, 2, 3

- after 2nd World War - aerial mapping

- after the "cold" World War - high resolution satellite mapping

- after the early developments of the mid 1980s - [terrestrial] mobile mapping

… AND 4!

- mapping from unmanned aerial systems

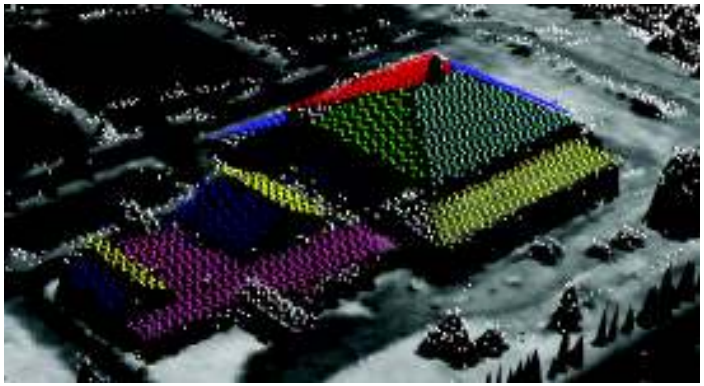
THE CURRENT TASKS & CHALLENGES OF RS&P

geometric & radiometric block adjustment of heterogeneous observations

- based on projective geometry
- based on radiative transfer models

- lead to “large” sparse systems of up to 100.000 unknows

non-semantic automatic feature extraction



semantic feature extraction

photogrammetric computer vision

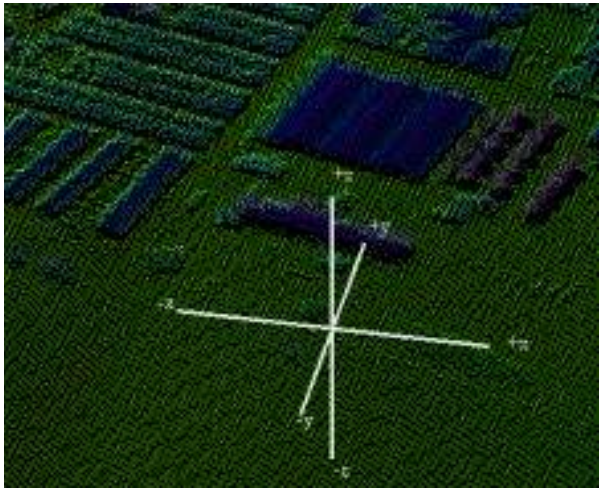
NON-SEMANTIC FEATURE EXTRACTION

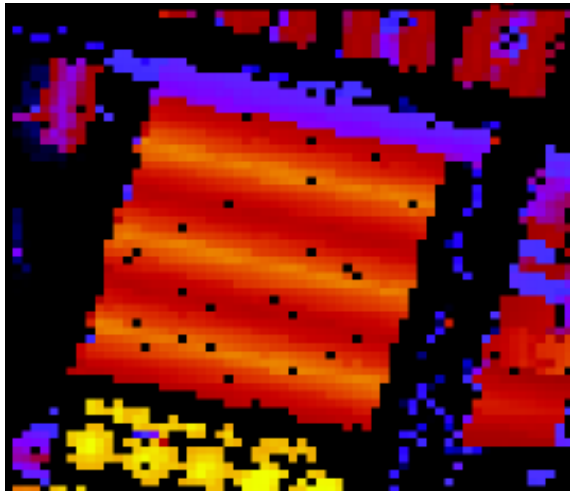
- RGB-to-LiDAR

  tie points,

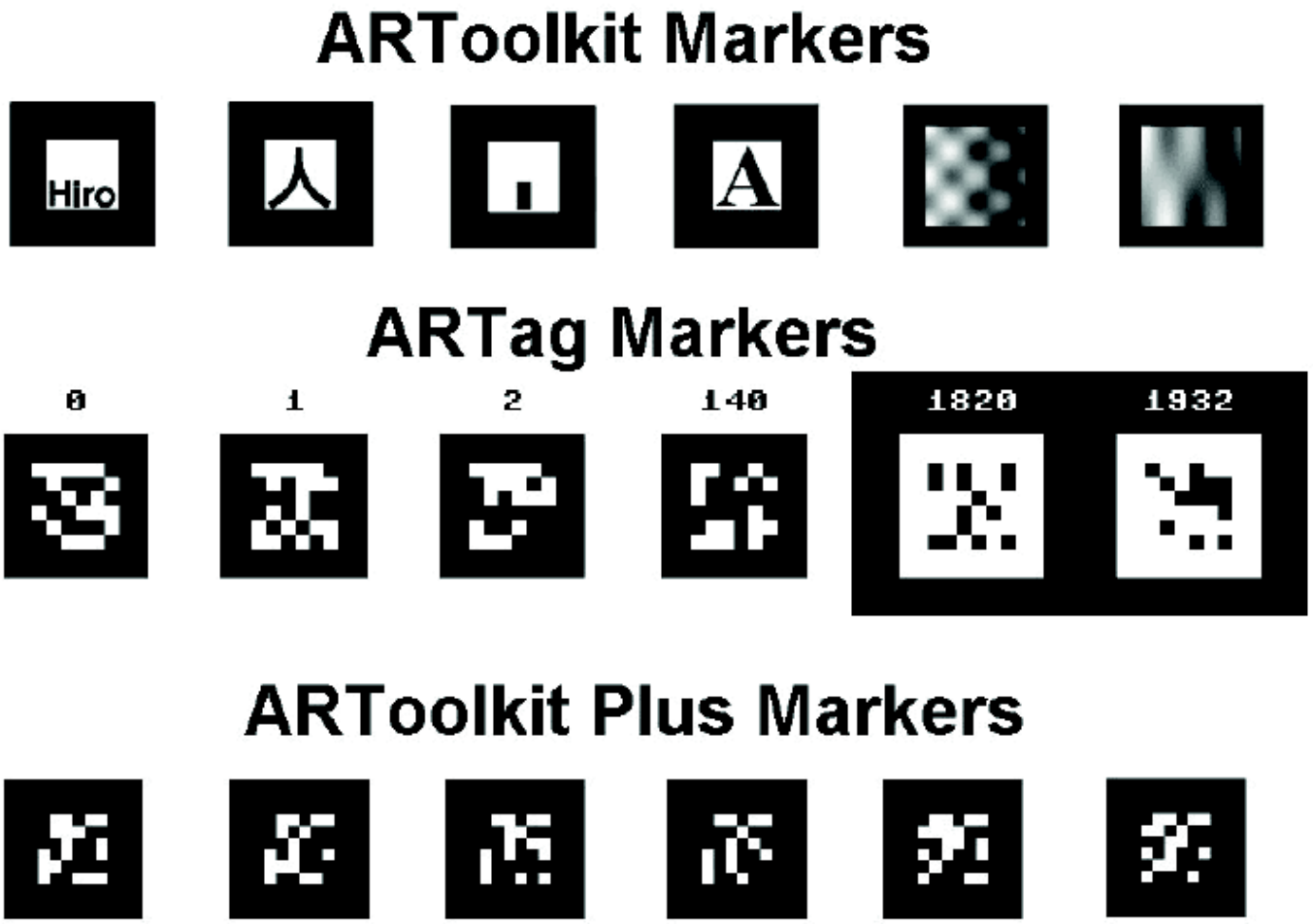
  tie lines,

  tie surfaces





ARTag MARKERS



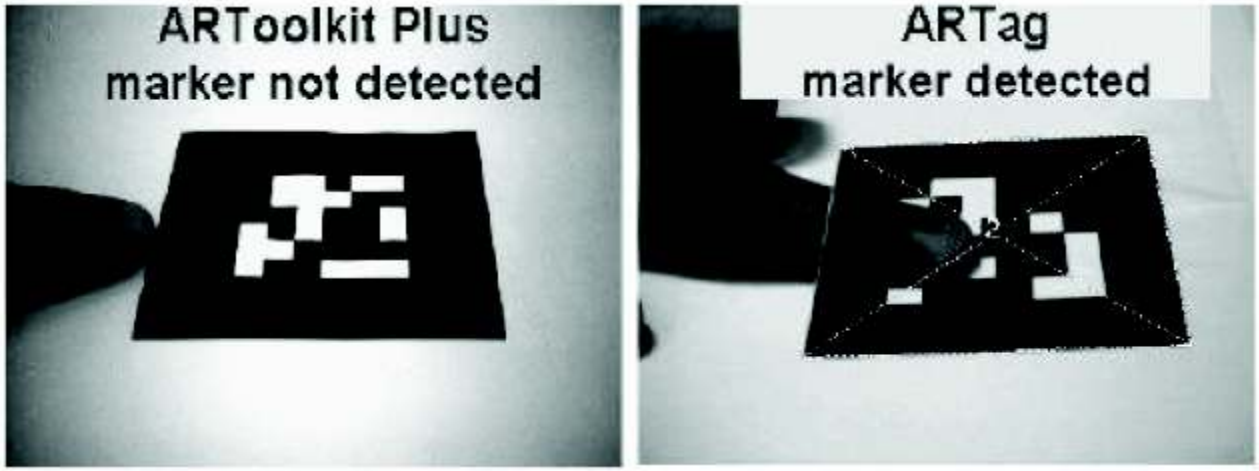
Fiala,M., 2005. ARTag rev. 2 fiducial marker system: vision-based tracking for AR. www.artag.net.

INTER-MARKER CONFUSION

2002 marks selected from 2^{36} codes

Hamming distance between markers:

```
D(110101011001100010011001101110101001,
  110101010001100000001101110111010001 01) = 5
```



CARTOGRAPHY / GEO-MODELING & GEO-VISUALIZATION

is the art, science and technology of making maps.

the sub-disciplines of cartography

- modeling (related to GIS: Geographic Information Systems)
  - map generalization
- visualization
  - symbology
  - related to computer graphics
- geographical names
- map projections

There is nothing more practical than a good theory.

– *James C. Maxwell (1831-1879)*